SHOWING THAT, at a particular point (a,b), the limit of $f(x,y)$ as $(x,y) \to (a,b)$ Does Not Exist

One way to show that a limit does not exist uses this fact:

<u>FACT</u>: Let (a,b) represent a particular point and $z = f(x,y)$ is a function of two variables.

<u>If</u> $\lim_{(x,y)\to(a,b)} f(x,y)$ Does Exist and $\lim_{(x,y)\to(a,b)} f(x,y) = L$,

<u>then</u>, for any two directed curves (a.k.a paths), curve C_1 and curve C_2, leading to the point (a,b),

$$\lim_{\substack{(x,y)\to(a,b)\\ \text{along curve } C_1}} f(x,y) = \lim_{\substack{(x,y)\to(a,b)\\ \text{along curve } C_2}} f(x,y) = L$$

<u>Resulting FACT</u>: <u>If</u> there are two curves C_1 and C_2 leading to the point (a,b) such that

$$\lim_{\substack{(x,y)\to(a,b)\\ \text{along curve } C_1}} f(x,y) = L_1 \quad \text{and} \quad \lim_{\substack{(x,y)\to(a,b)\\ \text{along curve } C_2}} f(x,y) = L_2 \quad \text{and } L_1 \neq L_2,$$

<u>then</u>

$\lim_{(x,y)\to(a,b)} f(x,y)$ <u>Does Not Exist</u>.

Using the Resulting Fact to prove that $\lim_{(x,y)\to(a,b)} f(x,y)$ Does Not Exist

Example: Let line ℓ_1 be the line in the (x,y)-plane with the equation $y = x$.

So, for any real number x , the point (x, x) is on line ℓ_1 .

Let line ℓ_2 be the line in the (x,y)-plane with the equation $y = -x$.

So, for any real number x , the point (x, – x) is on line ℓ_2 .

Let $z = f(x,y)$ be the function $z = f(x,y) = \dfrac{xy}{x^2+y^2}$. Let (a,b) = (0, 0), the origin.

We show that $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ *Does Not Exist* .

Let c_1 be any curve <u>along line ℓ_1</u> leading to (0, 0) .

For every point (x, y) on c_1 such that (x, y) ≠ (0, 0) , (x, y) = (x, x) and

$$f(x,y) = \frac{xy}{x^2+y^2} = \frac{x^2}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2}.$$

So, $$\lim_{\substack{(x,y)\to(0,0)\\ \text{along curve } c_1}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ \text{along curve } c_1}} \frac{xy}{x^2+y^2} = \frac{1}{2} = L_1 .$$

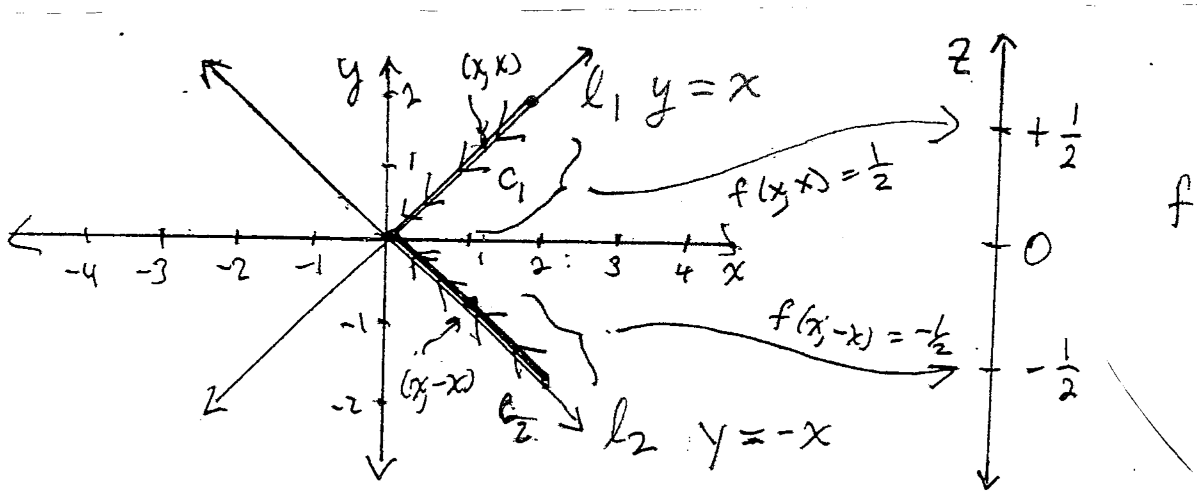
Let c_2 be any curve <u>along line ℓ_2</u> leading to (0, 0) .

For every point (x, y) on c_2 such that (x, y) ≠ (0, 0) , (x, y) = (x, – x) and

$$f(x,y) = \frac{xy}{x^2+y^2} = \frac{x(-x)}{x^2+x^2} = \frac{-x^2}{2x^2} = -\frac{1}{2}.$$

So, $$\lim_{\substack{(x,y)\to(0,0)\\ \text{along curve } c_2}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ \text{along curve } c_2}} \frac{xy}{x^2+y^2} = -\frac{1}{2} = L_2 :$$

Since $L_1 \neq L_2$, $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ = *Does Not Exist* .



$$f(x,y) = \frac{xy}{x^2+y^2}$$